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14. ABSTRACT
The Langlands Program was conceived initially as a bridge betwecn Number Thcory and Automorphic Representations, but it has now expanded into such areas as Geometry and Quantum Field Theory. Remarkably and surprisingly, the same salient features, such as the Langlands dual group, appear in these different contexts. Understanding the hidden patterns of the Langlands duality may thus hold the key to many important problems in modern mathematics and physics . One of the most exciting developments in this area in the last few years has been the work of Edward Witten and collaborators unifying the Langlands Program with dualities in quantum field theory and string theory. Specifically, they have related the so-called S-duality of the four-dimensional gauge theory to the geometric Langlands correspondence via the dimensional reduction from four to two dimensions. My work on this project continued this line of research as wcll as other topics on Langlands duality in Representation Theory, Geometry and Quantum Physics.

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FINAL REPORT FOR AFSOR GRANT FA9550-07-1-0543 GEOMETRIC LANGLANDS PROGRAM AND DUALITIES IN QUANTUM PHYSICS

EDWARD FRENKEL

1. Overview and results of previous support

The Langlands Program was conceived initially as a bridge between Number Theory and Automorphic Representations, but it has now expanded into such areas as Geometry and Quantum Field Theory. Remarkably and surprisingly, the same salient features, such as the Langlands dual group, appear in these different contexts. Understanding the hidden patterns of the Langlands duality may thus hold the key to many important problems in modern mathematics and physics.

One of the most exciting developments in this area in the last few years has been the work of Edward Witten and collaborators unifying the Langlands Program with dualities in quantum field theory and string theory. Specifically, they have related the so-called S-duality of the four-dimensional gauge theory (discovered by physicists P. Goddard, J. Nuyts and D. Olive in the 70s) to the geometric Langlands correspondence via the dimensional reduction from four to two dimensions. This successful influx of ideas from mathematics into physics was made possible in part due to the FAThM project.

With the support of the AFSOR grant, I have made further advances in this area. Here are the highlights:

(1) I have written the paper [8] with E. Witten on the geometric endoscopy and Mirror Symmetry, in which we have successfully used the newly found connection between the geometric Langlands correspondence and Mirror Symmetry of the Hitchin fibrations to gain new insights into the important mathematical problem of endoscopy. In particular, we have explained which A-branes on the Hitchin moduli space correspond to the B-branes supported at orbifold singular points of the dual moduli space. We have also observed some unexpected connections to the work of B.-C. Ngo on the fundamental lemma.

(2) In my joint work [7] with A. Losev and N. Nekrasov we have developed a novel approach to quantum field theories with instantons by using the "infinite radius limit" (rather than the limit of free field theory) as the starting point. We have analyzed the supersymmetric two-dimensional sigma models and four-dimensional Yang–Mills theory as full-feldged quantum field theories, beyond their topological sector. In particular, we defined the jet-evaluation observables and considered in detail their correlation functions. They are given by

regularized integrals over the moduli spaces of holomorphic maps, generalizing the Gromov–Witten invariants.

(3) I have developed (with C. Teleman and A.J. Tolland [11]) the theory of gauged Gromov-Witten invariants for algebraic varieties with torus action. We have introduced a geometric completion of the stack of maps from stable marked curves to the quotient stack $\text{pt}\,/GL_1$, and use it to construct some gauge-theoretic analogues of the Gromov-Witten invariants. We also indicate the generalization of these invariants to the quotient stacks $[X/GL_1]$, where X is a smooth proper complex algebraic variety.

(4) I have written the paper [1] with B. Feigin, in which we related the problem of quantization of classical soliton integrable systems, such as the KdV hierarchy, to an affine analogue of the Langlands duality. We have conjectured that common eigenvalues of the mutually commuting quantum Hamiltonians in a model associated to an affine Lie algebra should be encoded by affine opers associated to the Langlands dual affine Lie algebra. We have checked that our predictions are in agreement with the recent results by Bazhanov, Lukyanov and Zamolodchikov and others on the spectra of the quantum KdV Hamiltonians.

(5) In the joint papers [2, 3] with B. Feigin and L. Rybnikov, we have studied the spectra of the maximal commutative subalgebra of the universal enveloping algebra of a simple Lie algebra $\mathfrak{g}$, called the "shift of argument subalgebra". (We had earlier constructed this subalgebra using the center of the enveloping algebra of $\widehat{\mathfrak{g}}$ and its relation to opers of the Langlands dual group.) We have shown that generically their action on finite-dimensional modules is diagonalizable and their joint spectra are in bijection with the set of monodromy-free opers for the Langlands dual group ^LG of G on the projective line with regular singularity at one point and irregular singularity of order two at another point.

(6) In a series of papers [4, 5, 6] with D. Gaitsgory we have proved that the algebra of endomorphisms of a Weyl module of critical level is isomorphic to the algebra of functions on the space of monodromy-free opers on the disc with regular singularity and residue determined by the highest weight of the Weyl module; proved that the category of spherical $\widehat{\mathfrak{g}}$-modules of critical level is equivalent to the category of quasi-coherent sheaves on the ind-scheme of opers on the punctured disc which are unramified as local systems; and established an equivalence between a certain subcategory of $\widehat{\mathfrak{g}}$-modules of critical level and the category of quasi-coherent sheaves on the scheme of Miura opers for the Langlands dual group, thereby proving a conjecture from our earlier paper.

(7) I have proved (with my student X. Zhu [9]) that any flat G-bundle, where G is a complex connected reductive algebraic group, on the punctured disc admits the structure of an oper. This result is important in the local geometric Langlands correspondence proposed by Gaitsgory and myself.

(8) D. Hernandez and I have established [10] a correspondence (or duality) between the characters and the crystal bases of finite-dimensional representations of quantum groups associated to Langlands dual semi-simple Lie algebras.

In addition, I have organized (together with S. Gukov and D. Morrison) a three-week Mini-Program "Gauge Theory and Langlands Duality" at the Kavli Institute for Theoretical Physics in Santa Barbara, July 21–August 8, 2008. This Program brought together about 50 physicists and mathematicians to discuss the recent developments on the interface of the Langlands Program and Dualities in Physics. It was oriented towards physicists who wanted to learn this subject, including young researchers and graduate students. Among the topics were non-perturbative aspects of gauge theory, non-local operators, such as surface operators, 't Hooft and Wilson operators, and their action on D-branes. These ingredients, which play an important role in gauge theories, give valuable insights into the geometric Langlands Program, and conversely, mathematical ideas and constructions are beneficial for understanding quantum field theory.

The first week of this workshop included a tutorial intended for theoretical physicists. Lectures were given by P. Aspinwall, D. Ben-Zvi, E. Frenkel, S. Gukov, A. Kapustin, and D. Morrison. We had more than 15 graduate students, about half of whom were physicists and the other half – mathematicians. The second and third weeks were in the more traditional format of a workshop, with talks by such distinguished physicists as G. Moore and R. Dijkgraaf (talks are available online at the KITP Web site). Graduate students and postdocs ran their own seminar where they discussed the material in more detail. This Mini-Program has created a lot of interest in the subject, and we are now planning further activity in this area at KITP. Several excellent papers have already come out of this (see refs. [16–20]).

I have supported a postdoctoral fellow, Ghislain Fourier, in the Spring of 2008. He has written the papers [14, 15]. I have also supported two graduate students, A.J. Tolland and Xinwen Zhu (see [11] and [9]).

Here is the list of publications that resulted from this grant.

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